In *Journal of Philosophical Logic* vol. 41, no. 5 (2012): 841–51. Penultimate version.

Solving the color incompatibility problem

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It is commonly held that Wittgenstein abandoned the *Tractatus* largely because of a problem concerning color incompatibility. My aim is to solve this problem on Wittgenstein's behalf. First I introduce the central program of the *Tractatus* (§1) and the color incompatibility problem (§2). Then I solve the problem without abandoning any Tractarian ideas (§3), and show that given certain weak assumptions, the central program of the *Tractatus* can in fact be accomplished (§4). I conclude by distinguishing my system of analysis from others and by explaining the historical underpinnings of my understanding of the nature of elementary propositions (§5).

1 The central program of the *Tractatus*

The central program of the *Tractatus* is to show that we can completely analyze all ordinary language propositions. To make this precise, we need to say what propositions are, how to analyze them, and when an analysis of propositions counts as complete.

In (1929), Wittgenstein speaks interchangeably about analyzing propositions and analyzing "statements" such as 'it is eighty degrees outside' and 'E has two degrees of brightness' (167). This suggests that for Wittgenstein, propositions include utterances of ordinary sentences. Both primary and secondary literature on the color incompatibility problem suggests that the problem does not essentially concern utterances or sentences. One may just as well take the problem to be about the semantic values of sentences, i.e. about propositions in a relatively modern sense. To keep my discussion simple, I will take propositions to be sets of worlds. The ways in which genuine

1. Thanks to Bill Child, Ian Proops, Eric Swanson, Tim Williamson, and an anonymous reviewer for many helpful comments on earlier drafts of this paper.

Tractarian propositions differ from sets of worlds are incidental to my discussion of the color incompatibility problem.[2]

In order to analyze a proposition, we must find truth-functional combinations of propositions it is equivalent with. In particular, we can operate on propositions by taking their complements, unions, and intersections. By using these operations to combine some propositions to get another proposition, we are using the former propositions to analyze the latter.

The analysis of all ordinary language propositions counts as complete when we have found a set of *elementary* propositions (4.221). These elementary propositions must have two special features. First, any ordinary language proposition must be equivalent with some truth-functional combination of elementary propositions (4.51, 5, 5.01). Second, the elementary propositions must be logically independent from each other (5.134, 6.3751). That is, for any subset of elementary propositions, there must be a world where just those elementary propositions hold.

The central program of the *Tractatus* is to show that we can complete the analysis of ordinary language propositions into elementary propositions. Equivalently, the Tractarian program is to show that we can start with some set of logically independent propositions, and by repeatedly taking unions and intersections and complements of them, we can get any proposition expressed in ordinary language.

2 The color incompatibility problem

The problem for the *Tractatus* arises when we try to analyze propositions that say that a certain object has a certain color. The alleged problem is that it cannot be that all color propositions are combinations of logically independent propositions. Since we use ordinary language to express color propositions, this means that not all propositions expressed by ordinary language sentences can be combinations of logically independent propositions. So the Tractarian program cannot be accomplished.

The problem arises not just for color propositions, but for any propositions saying that a certain object has one of a number of mutually exclusive properties. Here is one canonical statement of the general incompatibility problem, from Hacker 1989:

> A proposition p attributing a degree of quality to an object, e.g. 'A is $5R$', is either elementary or compound. If p is compound then it must be a conjunction of elementary propositions each attributing a 'quantity' of R to A, conjunctively implying A's possession of $5R$. But this is not possible.

2. See §5 for detailed arguments concerning my understanding of the nature of elementary propositions.

> [And] if p is an elementary proposition [then] 'A is $5R$' and 'A is $6R$' must be logically related to each other even though they are elementary. (111)

Different scholars present different versions of the color incompatibility problem.[3] But the argument usually consists of the same two steps.

First, we note that there must be non-elementary color propositions. Color propositions are mutually exclusive. So if more than one color proposition were elementary, some elementary propositions would be mutually exclusive. But elementary propositions must be logically independent (e.g. "it is a sign of a proposition's being elementary that there can be no elementary proposition contradicting it" (4.211)).[4] For example, the proposition that A is $5R$ and the proposition that A is $6R$ cannot both be true. So these two propositions are not logically independent, and so they cannot both be elementary. More generally, there can be at most one elementary color proposition.

Second, we note that non-elementary color propositions cannot be truth-functional combinations of elementary propositions. For example, suppose that the proposition that A is $1R$ is the unique elementary color proposition. The problem is that there is no way to operate on this single elementary proposition with truth functions to get the non-elementary proposition that A is $5R$. The proposition that A is $5R$ is not simply the negation of the proposition that A is $1R$, for instance, or the conjunction of this elementary proposition with itself. And so the proposition that A is $5R$ is not a truth-functional combination of elementary propositions.

This is bad news for the Tractarian program. It appears that many color propositions must fail to be truth-functional combinations of elementary propositions, so we could not possibly complete the analysis of ordinary language propositions into logically independent components. Dale Jacquette summarizes the consequences:

> I believe as others have also suggested that there is strong evidence for the hypothesis that Wittgenstein's later philosophy was provoked by a realization that the early logical atomism of the *Tractatus* was finally incapable of dealing adequately with the color incompatibility problem. (354)

If we cannot analyze color propositions into logically independent components, then *a fortiori* we cannot completely analyze all propositions, and we must abandon the central program of the *Tractatus*.[5]

3. For recent discussions of the problem, see Jacquette 1990, Carruthers 1990, von Wright 1996, and Friedlander 2001.
4. *Tractatus* passages are quoted from the 1921 Pears-McGuinness translation.
5. This is a commonly accepted conclusion. For instance, Austin 1980, Pears 1987, Hacker 1989, and Sievert 1989 all agree that color incompatibility plays a central role in motivating Wittgenstein to abandon the Tractarian program.

3 Solving the color incompatibility problem

The first step of the color incompatibility argument is unproblematic. There must be non-elementary color propositions. But the second step fails. One can indeed combine logically independent propositions with truth functions to get any non-elementary color proposition. The key idea is that the elementary color propositions may be gerrymandered disjunctions of simpler-looking color propositions, while all simpler-looking propositions are themselves non-elementary.

For a toy example, suppose that there are four ordinary language color propositions, each of which is true at some world: that A is red, that A is yellow, that A is green, and that A is blue. Consider the following propositions:

(P_1) that A is red or yellow

(P_2) that A is red or green

These propositions are logically independent. Both are true when A is red, just P_1 is true when A is yellow, just P_2 is true when A is green, and neither is true when A is blue.

Moreover, each of our four ordinary language color propositions is a truth-functional combination of P_1 and P_2. The proposition that A is red is $(P_1 \wedge P_2)$. The proposition that A is yellow is $(P_1 \wedge \neg P_2)$. And so on. Hence the four mutually exclusive color propositions we started with do not pose any problem for the *Tractatus*. They are indeed truth-functional combinations of logically independent propositions.[6]

This strategy for constructing elementary propositions generalizes to the case of genuine color propositions. Let A be a particular object. In the standard statement of the color incompatibility problem, there are continuum-many propositions attributing a color to A, each of which is true at some world. There are also continuum-many infinitely long sequences of 0's and 1's. Let us fix a bijection between these sets, a mapping from possible colors of A to infinite binary strings. Consider the following set of propositions:

$$\{Q_i : \text{the color of } A \text{ is mapped to a string with a 1 in the } i\text{th place}\}$$

The Q_i propositions are logically independent. For instance, all of them are true when the color of A is mapped to the string of all 1's. Just Q_1 is true when the color of A is

6. Two authors mention disjunctive color propositions in connection with the color incompatibility problem. Carruthers 1990 notes that certain propositions attributing ranges of colors can be logically independent. But Carruthers says that the propositions he constructs ultimately cannot be elementary, and he concludes that "the attempt to extend our materialist model to include descriptions of colour has failed" (145). In (1996), von Wright notes that propositions like '*a* is bluish' and '*a* is reddish' may help analyze propositions like '*a* is violet' (16). But von Wright does not talk about solving the general color incompatibility problem. See §5 for further discussion.

mapped to the string of a 1 followed by infinitely many 0's. And so on. Whether the color of A is mapped to a string with a 1 in the first place is independent of whether the color of A is mapped to a string with a 1 in the second place. Since A could be any color, the color of A may be mapped to a string in which 1's occur in any combination of places. So any combination of the Q_i propositions may be true.

Moreover, each ordinary language color proposition is a truth-functional combination of the Q_i propositions. For example, take the proposition that A is red. Our color-to-sequence bijection maps the color red to some string r of 0's and 1's. The proposition that A is red is equivalent to the proposition that the color of A is mapped to r. And this proposition is in turn equivalent to a conjunction of propositions that say whether there is a 1 in each place of the string to which the color of A was mapped. But this is simply a conjunction of Q_i propositions and their negations. So the proposition that A is red is a truth-functional combination of Q_i propositions. The same holds for any color proposition we want to analyze. Hence color propositions do not pose any problem for the *Tractatus*. They are indeed truth-functional combinations of logically independent propositions, namely the Q_i propositions.

4 Conditions on accomplishing the *Tractatus* program

I have shown that the *Tractatus* program can be accomplished in a special case. The above strategy shows how for any 2^n mutually exclusive propositions, we can find n logically independent propositions such that the mutually exclusive propositions are truth-functional combinations of them. This works when n is any cardinal number. In the toy case, four color propositions are truth-functional combinations of two logically independent propositions. In the more general case, continuum-many color propositions are truth-functional combinations of countably many logically independent propositions.

But the general Tractarian program is not limited to analyzing mutually exclusive propositions. In full generality, the program is to show that we can analyze all propositions expressed in ordinary language. So in this section, I will show that any infinite number of propositions, standing in arbitrary relations of logical dependence, are truth-functional combinations of logically independent propositions.

The key idea is that propositions standing in arbitrary relations of logical dependence are themselves truth-functional combinations of mutually exclusive propositions. More precisely, given an arbitrary set of propositions, we can find 2^n (for some n) mutually exclusive propositions such that any proposition in our given set is a *conjunction* of some of them. In §3, I argued that we can always show that 2^n mutually exclusive propositions are truth-functional combinations of logically independent propositions. Since conjunctions of truth-functional combinations of some propositions are themselves truth-functional combinations of those propositions, this gives us our desired conclusion: that any propositions standing in arbitrary relations of logical dependence are truth-functional combinations of logically independent propositions.

It just remains to be shown that given an arbitrary set S of propositions, we can always find 2^n (for some n) mutually exclusive propositions such that any member of S is a conjunction of some of them. Consider the following equivalence relation R on worlds: let two worlds bear the R relation to each other just in case every member of S has the same truth value at those worlds. Let S' be the set of equivalence classes of worlds under the R relation. Intuitively, the members of S' represent possible combinations of truth values that the S propositions could have. Each S' proposition says exactly which S propositions are true and which are false, and nothing more.

The S' propositions are mutually exclusive. And each S proposition is a truth-functional combination of the S' propositions, namely the conjunction of all the S' propositions that entail it. So if there are 2^n propositions in S' then we are done: the S' propositions are 2^n mutually exclusive propositions such that any member of S is a conjunction of some of them. If S' does not contain 2^n propositions, then we may simply repeatedly replace any single member of S' with mutually exclusive propositions whose disjunction is that member. In this way, we can increase the cardinality of S' until it contains 2^n mutually exclusive propositions.

Once this is complete, S' is a set of 2^n mutually exclusive propositions, and every S proposition is a conjunction of S' propositions. Furthermore, the argument in §3 shows that there is a set of logically independent propositions such that each S' proposition is a truth-functional combination of them. The S propositions are conjunctions of truth-functional combinations of these logically independent propositions, which means the S propositions are themselves truth-functional combinations of logically independent propositions, as desired. Given an arbitrary set of propositions, we can construct a set of logically independent propositions such that the given propositions are truth-functional combinations of them.

There is one small caveat: I have assumed that given a set S' of propositions, we can construct a set of 2^n propositions by repeatedly replacing a member of S' with propositions whose disjunction is that member. In a small number of cases, we cannot do this and therefore cannot analyze a certain set of propositions. For example, suppose there are exactly three possible worlds, corresponding to three possible colors for A: red, yellow, and green. Then it is impossible to find logically independent propositions such that each color proposition is some truth-functional combination of them. In order to rule out this kind of defective counterexample, we need a weak assumption: that there is some power of 2 between [inclusively] the cardinality of the constructed set S' and the number of possible worlds. It suffices to assume that the number of worlds is a power of 2, since then in the worst case, we may replace members of S' with mutually exclusive propositions until S' contains just those 2^n mutually exclusive propositions that consist of a single world.

Once we make this weak assumption about the number of possible worlds, we can use the strategy I have outlined to accomplish the Tractarian program. Given any infinite number of propositions standing in arbitrary relations of logical dependence, there will always be a set of logically independent propositions such that the given propositions are truth-functional combinations of them. This result straightforwardly applies to the infinite number of propositions expressed by sentences of ordinary language. The central program of the *Tractatus* can in fact be accomplished.

5 Discussion

It is instructive to compare the present system of Tractarian analysis with the system proposed in Carruthers 1990. Carruthers observes that one might make some progress towards analyzing color propositions by matching each color with a real number in a grid. One could then introduce a candidate set of elementary propositions, namely the set of strings of five concatenated names, where 'abctn' states that a point-mass exists at the intersection of a, b, and c at time t, and that its color is matched with a number that has a '1' in the nth decimal place (144). But Carruthers succinctly explains why these propositions cannot really be the elementary ones:

> It is still the case that 'abctn' must entail 'abct' [i.e. that there is a point-mass at abct]. For if there is no point-mass there, then it cannot have a colour. Moreover, 'abctn' and 'not abctn' can both be false together (that is, if 'abct' is false), thus debarring them from being genuinely elementary, on the *TLP* account. (145)

To spell this out: the problem with the Carruthers system is that not every proposition is a truth-functional combination of the 'abctn' propositions. In particular, since each 'abctn' proposition and its negation entail that there is a point-mass at a certain location, there is no way to represent the proposition *that there is no point-mass at that location* as a truth-functional combination of the 'abctn' propositions. From another angle: the ingredients of the Carruthers system cannot distinguish between two distinct propositions, namely that a point-mass at a certain location has the '0' color, and that there is no point-mass at that location. The conjunction of the negations of all the 'abctn' propositions constitutes the former proposition, and so it cannot also constitute the latter. There are only exactly 2^n truth-functional combinations of the 'abctn' propositions available, and no extra combination is available to represent the absence of a point-mass.

This problem for the Carruthers system is not a problem for the system of analysis that I develop in §4. In contrast to the procedure that Carruthers gives, my §4 procedure can be used to analyze the $2^n + 1$ pairwise incompatible propositions that describe the color *or absence* of a point-mass at a certain location. This difference highlights a unique feature of my system of analysis: while my §3 discussion has much in common with Carruthers 1990, the procedure developed in §4 is much more powerful than the system of analysis that Carruthers develops. I have stated a procedure for analyzing an arbitrary number of propositions that stand in arbitrary relations of logical dependence. *A fortiori*, I have stated a procedure for analyzing an arbitrary number of pairwise incompatible propositions. The generality of the §4 procedure means that there can be no counterexamples to my analysis like 'there is no point-mass at such-and-such location' lurking in the wings.

One might have another sort of concern about the §4 procedure, though, namely that the gerrymandered S' propositions do not have the right form to be elementary ones. In particular, certain *Tractatus* passages suggest that elementary propositions consist of names of Simples, i.e. necessarily existing simple objects: "An elementary proposition consists of names. It is a nexus, a concatenation of names... A name means an object... Objects are simple... Objects are what is unalterable and subsistent" (4.22, 3.203, 2.02, 2.0271). If it is central to the early work of Wittgenstein that elementary propositions consist of names of Simples, then it might appear that one could object that my analysis does not accomplish the central program of the *Tractatus*.

However, there is much textual evidence to suggest that the early Wittgenstein understands requirements regarding elementary propositions and Simples so that they are compatible with the system of analysis I have given. For instance, it is true that predicates and relation symbols may play a role in representing the S' propositions, just as 'red' is used in representing the P_i propositions in §3. But Wittgenstein suggests that he understands his remarks in *Tractatus* 4.22 to be compatible with the claim that representations of elementary propositions may include not only strings of names such as 'abctn' but also predicates and relation symbols: "Names are the simple symbols: I indicate them by single letters ('x', 'y', 'z'). I write elementary propositions as functions of names, so that they have the form 'fx', 'ϕ(x,y)', etc." (4.24, cf. Wittgen-

STEIN 1916, p. 71e for related remarks).[7]

One might object that representations of the S' propositions include not only predicates and relation symbols, but also logical constants. For example, one uses disjunction to represent the P_i propositions in §3, and logical constants certainly do not appear in the examples of elementary proposition forms in *Tractatus* 4.24. But Wittgenstein suggests that even logical constants can appear in our representations of elementary propositions. In (1929), Wittgenstein tries to analyze simple-looking color propositions as logical combinations of complex-looking propositions. He then presents his own understanding of the color incompatibility problem for this strategy:

> If, on the other hand, we try to distinguish between the units and consequently write $E(2b) = E(b') \;\&\; E(b'')$, we assume two different units of brightness; and then, if an entity possesses one unit, the question could arise, which of the two—b' or b''—it is; which is obviously absurd. (33)

This passage shows that Wittgenstein does not reject his own gerrymandered analyses of color propositions because they contain logical constants. He rejects his attempted analyses because they do not work. This is evidence that Wittgenstein does not think that how we express a proposition in natural language is a reliable guide to whether it is a viable candidate for being an elementary proposition, in the sense of 'elementary' relevant to the central project of the *Tractatus*. Wittgenstein himself was not troubled by objections to his analysis regarding Simples and the form of elementary propositions, and this suggests that at least in 1929, Wittgenstein would have been sympathetic to my solution to the color incompatibility problem.[8]

Reflecting on more general themes in Wittgenstein's early work can help us better understand how Wittgenstein reconciles his remarks about Simples with his gerrymandered analyses of ordinary propositions. In discussions as early as the *Notebooks* and the *Tractatus*, Wittgenstein is attracted by the idea that both descriptions and objects are *simple* only relative to a system of description. For example, Wittgenstein talks about using a square mesh to describe a surface and then notes that using a triangular mesh might have made the description simpler (6.341). The mesh stands for any system of description: "The different nets correspond to different systems for describing the world. . . it can be described more simply with one system of mechanics than with another" (6.341, 6.342). An object itself is simple only relative to a conceptual scheme: "This object is *simple* for *me*!" (WITTGENSTEIN 1916, 70e).

7. PEARS 1987 suggests that one may reconcile *Tractatus* 4.22 and 4.24 by allowing that function symbols may occur in elementary propositions, while not allowing that they may occur *as names* (142).
8. This interpretation is consonant with assumptions of some secondary literature on the color incompatibility problem, see for instance HACKER 1989, p. 111; JACQUETTE 1998, p. 172ff.; and SOAMES 2003, pp. 237–238.

These passages suggest an explanation for why Wittgenstein is sanguine about using apparently complex propositions to analyze apparently simple ones. He refuses to derive any conclusions about simplicity from facts about how propositions appear in ordinary language. In this respect, the *Tractatus* anticipates Goodman 1955. Both Wittgenstein and Goodman deny that ordinary language color predicates are less disjunctive than other color predicates. The property of being grue appears gerrymandered relative to actual ordinary languages, and simple relative to others. Exactly the same can be said for the color properties ascribed by the propositions in §3 and for the S' propositions constructed in §4. It may be that logical constants must be used to represent elementary propositions in a regimented variant of ordinary language. But one could also simply implement the §4 procedure and then introduce a name for every mereological sum of objects described by some S' proposition, and a relation symbol for every relation ascribed by some S' proposition. In other words, one could introduce 'grue'-like predicates and relation symbols after completing the analysis of propositions in ordinary language. It would then be possible to express elementary propositions using only simple expressions of the constructed language. And relative to that language, the requirement that elementary propositions consist of names of Simples would be met.[9]

It is a familiar observation that Wittgenstein came to see simplicity as a language-relative notion in his later work, e.g. §39–64 of the *Investigations*. Carruthers notes that for the later Wittgenstein, "there are no such notions as absolute simplicity or absolute complexity. . . on the contrary, 'complex' means different things in different sorts of context, and in different language-games" (89). According to Wittgenstein 1951, even the simplicity of colors is relative to a language game: "What is there in favor of saying that green is a primary color, and not a mixture of blue and yellow?. . . here there are language-games that decide these questions" (§158). This discussion of primary colors reflects a deeply liberal attitude about what concepts may be the fundamental elements of our conceptual scheme. Though this passage is from a text composed in the last months of Wittgenstein's life, the seeds of the liberalism that Wittgenstein expresses here are scattered throughout his early and middle corpus. It is this liberalism that leaves room for the system developed in §4 to accomplish the central program of the *Tractatus*.

9. For instance, one could introduce the predicate 'redoryellow' after completing the analysis of the toy model of color propositions in §3. In the general case, this exercise enacts a suggestion that Wittgenstein makes after pointing out that some facts about fundamentality depend solely on notation (5.474), namely that "all that is required is that we should construct a system of signs with a particular number of dimensions—with a particular mathematical multiplicity" (5.475).

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